

SPACIOUS KNOTS

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ABSTRACT. We show that there are hyperbolic knots in the 3-sphere that Benjamini–Schramm converge to hyperbolic space, meaning that there are hyperbolic knot complements such that the points of large injectivity radius take up the bulk of the volume. This answers a question of Brock and Dunfield.

1. INTRODUCTION

Many knots in the 3-sphere have complement admitting a hyperbolic structure [15], and by Mostow–Prasad rigidity [11, 13] and the Gordon–Luecke theorem [3], the structure is a complete knot invariant. However, it is still not well understood what properties of the hyperbolic metric distinguish knot complements from other hyperbolic 3-manifolds.

In [14], properties of hyperbolic knot complements are investigated by studying geometric limits. If a manifold M is a geometric limit of knot complements, then there exist hyperbolic knots whose geometric properties are very close to those of M . In [14], it is shown that any one-ended hyperbolic 3-manifold with finitely generated fundamental group that embeds in \mathbb{S}^3 is a geometric limit of hyperbolic knot complements. The class of such 3-manifolds is surprisingly broad, and includes hyperbolic 3-space \mathbb{H}^3 itself. However, in [7], compact submanifolds of \mathbb{S}^3 are presented whose interiors cannot be homeomorphic to any geometric limit of hyperbolic knot complements. Thus although the class of geometric limits of knot complements is large, there are geometric and topological restrictions on the manifolds that appear. These are not well understood.

This paper continues the investigation of limits of hyperbolic knot complements, particularly those converging to \mathbb{H}^3 . Because \mathbb{H}^3 is a geometric limit of knot complements, for any $R > 0$ there exists a hyperbolic knot K and a point $x \in \mathbb{S}^3 - K$ with injectivity radius $\text{inj}_x(\mathbb{S}^3 - K)$ at least R . Since the injectivity radius of points in a tubular neighborhood of K can be arbitrarily small, this implies that in a sequence $\mathbb{S}^3 - K_n$ converging to \mathbb{H}^3 , the cusp $N(K_n)$ must be pushed further and further outside larger and larger balls in $\mathbb{S}^3 - K_n$.

For M any hyperbolic 3-manifold, recall that the R -thick part of M is defined to be $M^{\geq R} = \{x \in M \mid \text{inj}_x(M) \geq R\}$, and its complement, denoted $M^{< R}$ is the R -thin part. The result of [14] implies that for any R , there exists a knot K whose R -thick part is nonempty. However, that paper does not consider the size of the set of all such points.

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Brock and Dunfield [2] recently asked: For any R , does there exist a sequence of knots for which the proportion of the volume of the R -thin part tends to zero? That is, for fixed R , it is known that there are knots for which the R -thick part is non-empty. But does there exist a sequence of knots for which the R -thick part takes up larger and larger proportions of the volume? This is called *Benjamini–Schramm convergence to \mathbb{H}^3* .

In this paper, we answer Brock and Dunfield’s question in the affirmative.

Theorem 1.1. *Given $R > 0$ and $\epsilon > 0$, there is a knot $K \subset \mathbb{S}^3$ such that $M = \mathbb{S}^3 - K$ is hyperbolic and*

$$(1.1) \quad \frac{\text{vol}(\{x \in M \mid \text{inj}_x(M) \geq R\})}{\text{vol}(M)} = \frac{\text{vol}(M^{\geq R})}{\text{vol}(M)} > 1 - \epsilon.$$

The quantity $\text{vol}(M^{\geq R})/\text{vol}(M)$ will be called the *R -volume ratio*.

The proof of Theorem 1.1 is a modification of Brock and Dunfield’s [2] proof of the existence of closed integral homology spheres Benjamini–Schramm converging to \mathbb{H}^3 , with added ingredients from [5] and [6]. Performing $1/n$ -surgeries on the knots of Theorem 1.1 produces new examples of homology spheres Benjamini–Schramm converging to \mathbb{H}^3 .

2. THICK MAPPING TORUS

As in [2], we first build mapping tori satisfying (1.1).

Given a surface S and a homeomorphism $f: S \rightarrow S$, there is an associated mapping torus

$$M_f = (S \times [-1, 1]) / (x, -1) \sim (f(x), 1).$$

Proposition 2.1. *For any $R > 0$ and any $\epsilon > 0$, there is a closed surface Σ of genus $g = g(R, \epsilon)$, an even number of points $\{p_1, \dots, p_{2k}\} \subset \Sigma$, and a pseudo-Anosov map f on the punctured surface $S = \Sigma - \{p_1, \dots, p_{2k}\}$ such that the R -volume ratio of M_f satisfies*

$$\frac{\text{vol}(M_f^{\geq R})}{\text{vol}(M_f)} > 1 - \epsilon,$$

and f extends to a homeomorphism $\Sigma \rightarrow \Sigma$ taking each p_i to itself.

Note that M_f has rank-2 cusps corresponding to $\{p_1, \dots, p_{2k}\} \times [-1, 1] / \sim$, and so $\text{vol}(M_f^{< R}) > 0$. To make the volume ratio large, we will ensure that the R -thin part lies only in the cusps and takes up a small proportion of the volume.

Any rank-2 cusp of a hyperbolic 3-manifold N has a neighborhood isometric to the quotient \mathbf{T} of a horoball about infinity by a group of parabolic isometries isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The boundary $\partial \mathbf{T}$ of this quotient is naturally a flat torus. If the quotient \mathbf{T} embeds in N , we say that \mathbf{T} is a *horoball neighborhood* of the cusp. We also use this notation and terminology for embedded disjoint unions of such neighborhoods.

Lemma 2.2. *For any $Q > 0$, there is a surface S of genus $g = g(Q)$ with an even number of punctures, and a pseudo-Anosov map $\varphi: S \rightarrow S$ such that the Q -thin part $M_\varphi^{< Q}$ of M_φ consists only of disjointly embedded horoball neighborhoods of the cusps of M_φ corresponding to the punctures of S , and φ fixes each puncture of S .*

Proof. Let f be any pseudo-Anosov homeomorphism on a punctured surface Σ with more than one puncture. The mapping torus M_f is hyperbolic [12], and the number of closed geodesics in M_f of length at most $2Q$ is finite. These correspond to conjugacy classes $[\gamma_i]$ of elements of $\pi_1(M_f)$. Since $\pi_1(M_f)$ is residually finite [9], there is a finite-index normal subgroup of $\pi_1(M_f)$ that does not contain any γ_i . Let N be the corresponding finite cover of M_f . Its shortest geodesic has length at least $2Q$. Moreover, the fibration of M_f over \mathbb{S}^1 pulls back to a fibration of N .

Let \mathbf{T} be a horoball neighborhood of the cusps of N . Let $[\delta_j]$ be the conjugacy classes of elements of $\pi_1(N)$ corresponding to the Euclidean geodesic loops on $\partial\mathbf{T}$ whose length is less than $2Q$. Residual finiteness gives us a finite-index normal subgroup of $\pi_1(N)$ that contains none of the δ_j , and passing to the corresponding finite cover of N produces a fibered manifold N' whose Q -thin part consists only of disjointly embedded horoball neighborhoods of the cusps.

The monodromy of the fibration $F \rightarrow N' \rightarrow \mathbb{S}^1$ is a homeomorphism of the punctured surface F that permutes the punctures of F , so a finite power takes each puncture to itself. Let N'' be the corresponding finite cover of N' , so the monodromy μ of the fibration $F \rightarrow N'' \rightarrow \mathbb{S}^1$ extends to a homeomorphism fixing all the punctures of F .

It may be the case that the fiber of N' has an odd number of punctures (even if Σ were evenly punctured). If so, we pass to a further cover as follows. Let α and β be elements of $H_1(N''; \mathbb{Z})$ corresponding to distinct punctures of F . Let $\theta: H_1(N''; \mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ be the homomorphism taking α and β to 1 and killing the other generators of $H_1(N'')$. Since μ fixes all of the punctures of F , the kernel $\ker(\theta)$ of the induced map $\pi_1(N'') \rightarrow \mathbb{Z}/2\mathbb{Z}$ is μ -invariant. Let N''' be the cover of N'' corresponding to $\ker(\theta)$. Since $\ker(\theta)$ is μ -invariant, the fiber S of N''' is the cover of F corresponding to $\ker(\theta) \cap \pi_1(F)$, and thus has an even number of punctures, say $2k$. The monodromy φ of the fibration $S \rightarrow N''' \rightarrow \mathbb{S}^1$ fixes the punctures of S by construction, and so $N''' = M_\varphi$ is the desired manifold. \square

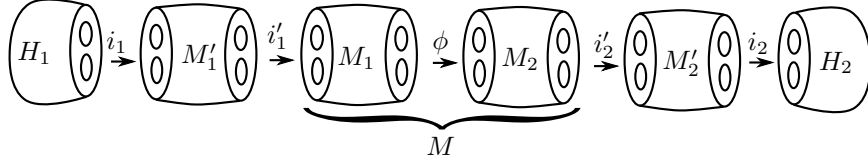
Proof of Proposition 2.1. Let $R > 0$ and $1 > \epsilon > 0$. Set $Q = 2R/\epsilon$, so $Q > R$. Lemma 2.2, applied to Q , gives a mapping torus M_φ with Q -thin part a horoball neighborhood of the cusps. Since $Q > R$, we have $M_\varphi^{<R} \subset M_\varphi^{<Q}$ and so the R -thin part is also a horoball neighborhood of the cusps.

Let \mathbf{T}_Q be a component of $\overline{M_\varphi^{<Q}}$, with $\mathbf{T}_R \subset \mathbf{T}_Q$ a component of $\overline{M_\varphi^{<R}}$. Consider the universal cover \mathbf{B}_Q of \mathbf{T}_Q as a horoball centered at infinity in the upper half-space model; similarly $\mathbf{B}_R \subset \mathbf{B}_Q$ is a horoball. The boundary $\partial\mathbf{B}_R$ is a horosphere of some height h_R . Let Δ be the $\mathbb{Z} \oplus \mathbb{Z}$ group of parabolics such that $\mathbf{B}_Q/\Delta = \mathbf{T}_Q$. The shortest hyperbolic translation distance of Δ acting on $\partial\mathbf{B}_R$ is R , and on $\partial\mathbf{B}_Q$ is Q . It follows that $\partial\mathbf{B}_Q$ has height $\epsilon h_R/2$.

Now, the volume of a horoball neighborhood of a cusp is half the area of its boundary, and so the volume ratio $\text{vol}(\mathbf{T}_R)/\text{vol}(\mathbf{T}_Q) = \text{area}(\partial\mathbf{T}_R)/\text{area}(\partial\mathbf{T}_Q)$. The heights of the respective horospheres imply the area ratio is $\epsilon^2/4$.

Choosing Q in this way guarantees that

$$\frac{\text{vol}(M_\varphi^{<R})}{\text{vol}(M_\varphi^{<Q})} < \frac{\epsilon^2}{4}.$$

FIGURE 1. Gluing all pieces yields \mathbb{S}^3

Then

$$\frac{\text{vol}(M_\varphi^{\geq R})}{\text{vol}(M_\varphi)} = 1 - \frac{\text{vol}(M_\varphi^{< R})}{\text{vol}(M_\varphi^{\geq Q} + M_\varphi^{< Q})} > 1 - \frac{\text{vol}(M_\varphi^{< R})}{\text{vol}(M_\varphi^{< Q})} > 1 - \epsilon^2. \quad \square$$

3. THE TOPOLOGICAL CONSTRUCTION

Our knot complements will be built from several geometric pieces.

Let $R, \epsilon > 0$ and let $g(R, \epsilon)$ be as in Proposition 2.1. Consider the Heegaard splitting of \mathbb{S}^3 of genus $g(R, \epsilon)$ with Heegaard surface Σ . So

$$\mathbb{S}^3 \cong (H'_1 \cup_\Sigma H'_2)/\psi,$$

with H'_1, H'_2 handlebodies, $\partial H'_i \cong \Sigma$, and $\psi: \Sigma \rightarrow \Sigma$ a homeomorphism.

Consider a regular neighborhood of Σ in \mathbb{S}^3 homeomorphic to $\Sigma \times [-1, 1]$ made of two pieces $M'_1 \cong \Sigma \times [-1, 0] \subset H'_1$ and $M'_2 \cong \Sigma \times [0, 1] \subset H'_2$ glued together by $\psi: \Sigma \times \{0\} \rightarrow \Sigma \times \{0\}$.

Let $H_1 = H'_1 - M'_1$ and $H_2 = H'_2 - M'_2$. We then obtain \mathbb{S}^3 by gluing four pieces: H_1, M'_1, M'_2 , and H_2 . Denote the gluing map between H_1 and M'_1 by $i_1: \partial H_1 \rightarrow \Sigma \times \{-1\}$, and that between M'_2 and H_2 by $i_2: \Sigma \times \{1\} \rightarrow \partial H_2$.

We further split each of M'_1 and M'_2 into two pieces, as follows. Let $M'_1 = \Sigma \times [-1, -\frac{1}{2}]$ and let $M_1 = \Sigma \times [-\frac{1}{2}, 0]$, with gluing map $i'_1: \Sigma \times \{-\frac{1}{2}\} \rightarrow \Sigma \times \{-\frac{1}{2}\}$, and let $M_2 = \Sigma \times [0, \frac{1}{2}]$ and $M'_2 = \Sigma \times [\frac{1}{2}, 1]$, with gluing map $i'_2: \Sigma \times \{\frac{1}{2}\} \rightarrow \Sigma \times \{\frac{1}{2}\}$.

We let $M = (M_1 \cup M_2)/\psi$.

This is all illustrated schematically in Figure 1.

4. GEOMETRIC TANGLES

We build our knots by gluing together tangles in H_1, M'_1, M, M'_2 , and H_2 . The pieces H_1, M, H_2 and their tangles are chosen first and are then fixed throughout the construction.

Start with the Heegaard splitting of \mathbb{S}^3 from the previous section. Let K be any closed curve in \mathbb{S}^3 meeting the Heegaard surface Σ transversely in $2k$ points $\{p_1, \dots, p_{2k}\}$. Recall that $M'_1 \cup M \cup M'_2$ is homeomorphic to $\Sigma \times [-1, 1]$, a regular neighborhood of Σ in \mathbb{S}^3 . We homotope K so that it meets this neighborhood in arcs of the form $p_j \times [-1, 1]$. We will homotope K relative its intersection with $\partial H_1, \partial M'_1, \partial M, \partial M'_2$, and ∂H_2 so that its intersections with each of H_1, M'_1, M, M'_2 , and H_2 are of a particularly nice form.

The following lemma follows easily from the work in [5].

Lemma 4.1. *Fix $\delta > 0$. Let N be a compact orientable 3-manifold. Let A be a 1-manifold properly embedded in N such that each component of $\partial N - \partial A$ has negative Euler characteristic, and let P be a pants decomposition of the punctured*

surface $\partial N - \partial A$. Then there is a 1-manifold B homotopic to A relative to its endpoints such that

- (1) $N - B$ is hyperbolic with totally geodesic boundary,
- (2) P has length less than δ on the boundary of $N - B$, and
- (3) all arcs in B correspond to rank-1 cusps.

Moreover, the manifold $N_{B,P} = N - (B \cup P)$ admits a hyperbolic structure with totally geodesic boundary a collection of 3-punctured spheres.

Remark on the proof. When A has empty boundary, parts (1)–(3) of the Lemma are explicit in [5], and straightforward modifications to the argument there produces the proof in the general case.

Note that the complement of a disjoint union of simple geodesics in a hyperbolic 3-manifold admits a complete hyperbolic structure [8]. So the manifold obtained by removing P from the double $D(N - B)$ of $N - B$ admits a complete hyperbolic structure. Moreover, this manifold is the double $DN_{B,P}$ of $N_{B,P}$, and it follows that $N_{B,P}$ admits a hyperbolic structure with totally geodesic boundary a collection of 3-punctured spheres. \square

If α is a curve on the boundary T of a cusp neighborhood, then the normalized length of α is defined to be

$$\mathcal{L}(\alpha) = \frac{\text{length}(\alpha)}{\sqrt{\text{area}(T)}}.$$

Lemma 4.2. *Let F be an orientable surface of finite type. Let $L > 0$. Then there is a $\delta = \delta(\chi(F), L) > 0$ such that the following holds. Let N be an orientable finite-volume 3-manifold with totally geodesic boundary in which the length of a pants decomposition P is less than δ . Then $N' = N - P$ admits a finite-volume hyperbolic metric with totally geodesic boundary a collection of 3-punctured spheres, and, whenever N' is embedded isometrically into a complete hyperbolic manifold N'' with $\partial N'' = \emptyset$, the normalized length of each meridian corresponding to a curve of P in N'' is at least L .*

Proof. Let P_0 be a curve in P and let \mathbf{T}_0 be the corresponding cusp neighborhood in N'' . The meridian corresponding to P_0 is the filling slope on $\partial \mathbf{T}_0$ whose homology class dies under the projection homomorphism $H_1(\mathbf{T}_0) \cong \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} = \langle [P_0] \rangle$.

Let \mathcal{A} be the horospherical annulus $\partial \mathbf{T}_0 \cap N'$. It is shown in the proof of Theorem 39 of [6] that there is a $\delta = \delta(\chi(F), L) > 0$ such that the conformal modulus of \mathcal{A} is at least L provided that the length of P is less than δ in the totally geodesic boundary of N . This implies that the normalized length of the meridian at P_0 is at least L . \square

5. GEOMETRIC KNOTS

Our knot in \mathbb{S}^3 will be constructed by gluing together tangles in H_1 , M'_1 , M , M'_2 , and H_2 . The tangles in M'_1 and M'_2 will be braids provided by Proposition 2.1.

For fixed $R > 0$ and $\epsilon > 0$, Proposition 2.1 provides a mapping torus M_f whose R -volume ratio is at least $1 - \epsilon$. Take Σ , $\{p_1, \dots, p_{2k}\}$, $S = \Sigma - \{p_1, \dots, p_{2k}\}$, and f as in that proposition. Let M_f^∞ denote the infinite cyclic cover of M_f corresponding to the fiber, and let M_f^n denote the mapping torus corresponding to f^n . Finally, let P be any pants decomposition of S .

Given natural numbers m and n , there exists a maximally cusped hyperbolic structure on $S \times \mathbb{R}$ with curves $f^{-m}(P)$ corresponding to rank-1 cusps on one end, and $f^n(P)$ corresponding to rank-1 cusps on the other, by Thurston's Geometrization Theorem [16, 10, 12]. Let $S(f^{-m}(P), f^n(P))$ be this hyperbolic manifold.

The following lemma is almost identical to [2, Proposition 3.4], only we allow additional cusps corresponding to the punctures in our surface. It may be proven just as in that paper, using the fact that the Drilling Theorem applies to manifolds with additional cusps.

Lemma 5.1. *The maximally cusped structures $S(f^{-m}(P), f^n(P))$ for $m, n > 0$ converge strongly to the infinite cover M_f^∞ as $m, n \rightarrow \infty$. The manifolds $S(P, f^n(P))$ converge strongly to a manifold S_A with a degenerate end asymptotically isometric to the positive end of M_f^∞ and whose convex core boundary is a surface with parabolic locus P . The analogous statement holds for $S(f^{-n}(P), P)$. \square*

We now prove Theorem 1.1.

Proof of Theorem 1.1. For fixed R and ϵ , take Σ , $\{p_1, \dots, p_{2k}\}$, S , and f as in Proposition 2.1. Take a Heegaard splitting of \mathbb{S}^3 with Heegaard surface Σ , and H_1 , M'_1 , M , M'_2 , and H_2 as in Section 3. Finally, take $K \subset \mathbb{S}^3$ as in Section 4, meeting the Heegaard surface Σ for \mathbb{S}^3 transversely in points $\{p_1, \dots, p_{2k}\}$, and let P be any pants decomposition of S . Let $\delta > 0$.

Using Lemma 4.1, we homotope K relative its intersection with ∂H_1 and ∂H_2 so that $H_i - K$ has a hyperbolic structure with totally geodesic boundary in which the length of P is less than δ . Then $\overline{H}_i = H_i - (K \cup P)$ has a hyperbolic structure with totally geodesic boundary a collection of 3-punctured spheres, for each i .

Now we consider $M'_1 \cong S \times [-1, -\frac{1}{2}]$ and $M'_2 \cong S \times [\frac{1}{2}, 1]$. For any integer $r > 0$, the manifold $S \times [-1, -\frac{1}{2}] - (P \times \{-1\} \cup f^r(P) \times \{-\frac{1}{2}\})$ has hyperbolic structure with totally geodesic boundary, isometric to the convex core of $S(P, f^r(P))$. Similarly, $S \times [\frac{1}{2}, 1] - (f^r(P) \times \{\frac{1}{2}\} \cup P \times \{1\})$ has hyperbolic structure with totally geodesic boundary, isometric to the convex core of $S(P, f^r(P))$.

Consider $M = \Sigma \times [-\frac{1}{2}, \frac{1}{2}]$. Let A be the intersection of K and M . Let B be the tangle homotopic to A making the lengths of $P \times \{-\frac{1}{2}\}$ and $P \times \{\frac{1}{2}\}$ less than δ in $M - B$, given by Lemma 4.1. The manifold $M_{B,P} = M - (B \cup P \times \{\pm\frac{1}{2}\})$ admits a hyperbolic structure with totally geodesic boundary a collection of 3-punctured spheres. The map f^r extends to a self homeomorphism of $M = \Sigma \times [-\frac{1}{2}, \frac{1}{2}]$. This homeomorphism takes B to a collection of embedded arcs $f^r(B)$ in $f^r(M)$, and takes $P \times \{\pm\frac{1}{2}\}$ to $f^r(P) \times \{\pm\frac{1}{2}\}$.

Glue \overline{H}_1 and $\text{core}(S(P, f^r(P)))$ together via i_1 . Note this map takes P to P and takes 3-punctured spheres to 3-punctured spheres. Because there is a unique hyperbolic structure on a 3-punctured sphere, this extends to an isometry. Similarly, i'_1 induces an isometry from the right side of $\text{core}(S(P, f^r(P)))$ to the left side of $f^r(M)$, i'_2 induces an isometry from the right side of $f^r(M)$ to the left side of $\text{core}(S(f^r(P), P))$, and i_2 induces an isometry from the right side of $\text{core}(S(f^r(P), P))$ to \overline{H}_2 .

With these gluing maps, let

$$N_r = \overline{H}_1 \cup \text{core}(S(P, f^r(P))) \cup f^r(M_{B,P}) \cup \text{core}(S(f^r(P), P)) \cup \overline{H}_2.$$

By construction, N_r is homeomorphic to \mathbb{S}^3 with a link removed. One component of this link is K , and the other components are copies of the components of P and $f^r(P)$. We call the cusps corresponding to the latter curves the *horizontal* cusps.

The hyperbolic structures on \overline{H}_1 , \overline{H}_2 , and M are fixed, and so are their volumes. By Mostow–Prasad rigidity, the manifold $f^r(M)$ is isometric to M . It follows that the R -volume ratios of the links N_r approach the R -volume ratio of M_f as r goes to infinity. In particular, we may take r such that the ratio is at least $1 - \epsilon/2$.

To finish the proof, we fill the horizontal cusps of our links to obtain the desired knots. To do this, we use the Universal Hyperbolic Dehn Filling Theorem of Hodgson and Kerckhoff [4] and the Drilling Theorem of Brock and Bromberg [1]. Together, these theorems provide a universal L such that if the normalized length of each component of a Dehn filling slope is at least L , then the ϵ_3 -thick part of the filled manifold is $(1 + \epsilon/100)$ -bilipschitz to that of the original manifold, where ϵ_3 is the 3-dimensional Margulis constant.

In our case, we would like to fill our link complement along all of the horizontal meridians. By Lemma 4.2, there is a δ such that the normalized lengths of these meridians will be at least L provided the length of P is less than δ , and so we choose δ in this way.

Thus the ϵ_3 -thick part of the knot complement obtained by filling along the horizontal meridians is $(1 + \epsilon/100)$ -bilipschitz to the ϵ_3 -thick part of our link complement. Since the volume of the thin parts decrease under filling, the resulting knot is the desired one. \square

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